Periodic orbits of a one dimensional non autonomous Hamiltonian system

J.Bellazzini, V.Benci, M.Ghimenti

Dipartimento di Matematica Applicata
Università di Pisa
Via Bonanno Pisano 25/b 56126 PISA -Italy

Abstract

In this paper we study the properties of the periodic orbits of $\ddot{x} + V'_x(t,x) = 0$ with $x \in S^1$ and $V'_x(t,x)$ a T_0 periodic potential. Called $\rho \in \frac{1}{T_0}\mathbb{Q}$ the frequency of windings of an orbit in S^1 we show that exists an infinite number of periodic solutions with a given ρ. We give a lower bound on the number of periodic orbits with a given period and ρ by means of the Morse theory.

Key Words: Morse theory, periodic orbits, twisting number

1 Introduction

In this paper we study the second order Hamiltonian system

$$\ddot{x} + V'_x(t,x) = 0 \tag{1}$$

where $x \in S^1 = \mathbb{R}/\mathbb{Z}$ and $V \in C^2(\mathbb{R} \times S^1)$ is a periodic potential with minimal period T_0.

There are two question that we study in this paper. First, we study the existence of periodic solutions of (1) in any connected component of the space of periodic trajectories, i.e in the space of trajectories that makes k_1 windings in S^1 in k_2T_0 time with k_1 and k_2 arbitrary integers that are coprime.

Second, called $\rho(x) = \frac{k_1}{k_2T_0}$ the frequency of windings of $x(t)$ in S^1, we study the existence of orbits with the same ρ that are not k_2T_0 periodic, i.e periodic orbits that make mk_1 windings in mk_2T_0 time $m \in \mathbb{N}$, with k_1 and k_2 coprime, when the solutions are not k_2T_0 periodic.

The problem of the search of periodic orbits is classical and a standard approach to these problems is that of studying the critical points of the action functional

$$f(x) = \frac{1}{k_2 T_0} \int_0^{k_2 T_0} \left(\frac{1}{2} |\dot{x}|^2 - V(t,x) \right) dt \tag{2}$$

in the space of functions that makes k_1 windings in $k_2 T_0$ time.

This method have been largely used by many people in the last twenty years: see e.g. the book of Rabinowitz [13] and the references therein.

The problem of the search of periodic orbits with a given frequency of windings in S^1 is closely related to that of the existence of subharmonic orbits. In particular, for what concern the existence of subharmonic orbits, we quote [14] for a general Hamiltonian system on $S^1 \times T^{2n}$ under non-degeneracy condictions, [9] for the existence of infinitely many subharmonics for more general Lagrangian systems and [15] for the case of a second order differential equation in $\mathbb{R}^n$ with a time dependent periodic potential and a periodic forcing term with zero mean value.

In his paper, the existence of periodic orbits of equation (1) has been studied by means of the Morse relations applied to the action functional. We will show that the Morse relations allows to prove the existence of infinitely many orbits with a given $\rho = \frac{k_1}{k_2 T_0}$ and to give a lower bound on the number of periodic orbits with a given period.

2 Statements of the results

We set

$$C^2_{k_1,k_2T_0} = \{[x] : x \in C^2(\mathbb{R},\mathbb{R}),\ x(t + k_2 T_0) = x(t) + k_1\};$$

where $[x] = x \mod 1$; namely $C^2_{k_1,k_2T_0}$ is the space of the periodic C^2-functions which make k_1 windings in $k_2 T_0$ time; thus we have that $C^2_{k_1,k_2T_0} \subset C^2_{mk_1,mk_2T_0}$, $m \in \mathbb{N}^+$. Given a periodic orbit, $x(t)$, the rotation frequency $\rho = \rho(x)$ associated to $x(t)$ can be defined as the frequency of windings of the periodic orbit in S^1, i.e. the number of windings divided by $k_2 T_0$. Given $k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{N}$, the periodic orbit that makes k_1 windings in $k_2 T_0$ time has a rotation frequency $\rho = \frac{k_1}{k_2 T_0}$. Clearly, the set of periodic functions with rotation frequency $\rho = \frac{k_1}{k_2 T_0}$ when k_1 and k_2 are coprime is given by $\bigcup_{m=1}^{\infty} C^2_{mk_1,mk_2T_0}$.

Definition 1. A periodic solution $x(t)$ of rotation frequency $\rho = \frac{k_1}{k_2 T_0}$ with k_1 and k_2 coprime is called a *fundamental solution* if $x \in C^2_{k_1,k_2T_0}$. Otherwise, if $x \notin C^2_{k_1,k_2T_0}$, it is called *non-fundamental*.

Definition 2. A periodic solution $x(t)$ is called *non-resonant* if the linearized equation $\ddot{y} + V''(t, x(t))y = 0$ has no periodic solution (different from 0); the equation (1) is called *non-resonant* if all its periodic solutions are non-resonant.

From now on we assume that eq.(1) is non-resonant. This is a technical assumptions which makes easier to use Morse theory. In fact, if x is a non-resonant T-periodic solutions, it is a non-degenerate critical point of (2).

The first result of this paper is the following

Theorem 3. *If eq. (1) is non-resonant, for every $\rho = \frac{k_1}{k_2 T_0}$, with k_1 and k_2 coprime, equation (1) has exactly $2r$ fundamental solutions with $r > 0$ and infinitely many non-fundamental solutions.*

Clearly any k_2T_0-periodic solution $x(t)$ is also a mk_2T_0-periodic solution, $m \in \mathbb{N}$; thus $x(t)$ is a critical point of the functional (2) with $T = k_2T_0$ and $T = mk_2T_0$ respectively, and the Morse index $m(x, T)$ is well defined for such values of T. Given a periodic orbit $x(t)$, we define the twisting frequency (also called the twisting number or the mean index) τ as the mean Morse index, i.e. $\lim\limits_{T\to\infty} \frac{m(x,T)}{T}$.

The second result of this paper concerns the number of non-fundamental solutions in $C^2_{pk_1,pk_2T_0}$ with p prime.

We introduce two function $\nu(\tau, \rho)$ and $\eta(\tau, \rho)$, that are related to the number of fundamental solutions with twisting frequency less than τ and rotation frequency ρ. The value of such functions permits to give a lower bound on the number of non-fundamental orbits with rotation frequency ρ in $C^2_{pk_1,pk_2T_0}$.

In order to state the main theorem we need to to classify the periodic orbits in two classes, the class α of periodic orbits with even Morse index and the class β with odd Morse index.

As we will see, the α-periodic orbits are those with distinct positive Floquet multipliers while the β-periodic are those possessing negative or complex Floquet multipliers. We set for any $\tau \in \mathbb{R}^+$ and $\rho \in \frac{1}{T_0}\mathbb{Q}$

$$\begin{aligned}
n_\alpha(\tau,\rho) &= \left\{ \begin{array}{c} \text{number of fundamental solutions } x \text{ of type } \alpha \text{ with} \\ \tau(x) = \tau;\ \rho(x) = \rho \end{array} \right\} \\
n_\beta(\tau,\rho) &= \left\{ \begin{array}{c} \text{number of fundamental solutions } x \text{ of type } \beta \text{ with} \\ \tau(x) = \tau;\ \rho(x) = \rho \end{array} \right\}
\end{aligned}$$

we can define the function

$$\chi(\tau, \rho) := n_\alpha(\tau, \rho) - n_\beta(\tau, \rho)$$

and the multiplicity functions

$$\nu(\tau,\rho) := \sum_{\zeta\le\tau}\chi(\zeta,\rho); \qquad \eta(\tau,\rho) := \sum_{\zeta<\tau}\chi(\zeta,\rho).$$

By this functions we can prove the second results of this paper

Theorem 4. *Let $\rho = k_1/(k_2T_0)$ with k_1 and k_2 coprime, p prime, and assume that eq.(1) is non-resonant.*

Then, the periodic solutions in $C^2_{pk_1,pk_2T_0}$ having rotation frequency ρ, and Morse index $2n$ are of type α and are at least

$$\nu\left(\frac{2n}{pk_2T_0},\rho\right) \quad \mod p.$$

Moreover, the periodic solutions in $C^2_{pk_1,pk_2T_0}$ having rotation frequency ρ, and Morse index $2n+1$ are of type β and are at least

$$-\eta\left(\frac{2n+2}{pk_2T_0},\rho\right) \quad \mod p.$$

As a consequence, we have the following corollary.

Corollary 5. *We set*

$$\Sigma = \overline{\{(\tau,\rho)\ :\ \nu(\tau,\rho)\neq 0\}}. \tag{3}$$

Then, for any $(\tau,\rho)\in\Sigma$, there exists a sequence of non-fundamental solutions $\{x_n\}$ of type α and a sequence of solution $\{y_n\}$ of type β such that

$$\begin{array}{ll} \rho(x_n)\to\rho & \tau(x_n)\to\tau \\ \rho(y_n)\to\rho & \tau(y_n)\to\tau \end{array}$$

3 The Morse relations

In order to obtain some estimates on the number of critical point, we must recall some features of Morse theory. After a short summary of the main results, we show some preliminary lemma useful to apply Morse theory to our framework. For an exhaustive treatment of Morse theory, and for the proofs of the results here collected the reader can check [6], [12], [3, 4, 5].

Definition 6. Let M a C^2 complete differential manifold and let $f\in C^2(M,\mathbb{R})$ function. Let $x\in M$ a critical point of f. Suppose that x is non-degenerate, i.e. the Hessian determinant does not vanish in x.

Then the Morse index $m(x)$ is the signature of the Hessian of f at x

By this definition it is possible to prove the following theorem.

Theorem 7. *Let M be a complete C^2 Riemannian manifold, $f \in C^2(M, \mathbb{R})$. Set*

$$f^b = \{x \in M \ : \ f(x) \leq b\}; \tag{4}$$
$$f_a^b = \{x \in M \ : \ a \leq f(x) \leq b\}; \tag{5}$$

let $c \in \mathbb{R}$ be the unique critical level in the interval $[a, b]$. Suppose that the critical points in $f^{-1}(c)$ are non-degenerate and suppose that f_a^b is a compact set in M. If there are n critical points of index q at level c, then

$$\dim H_q(f^b, f^a) = n, \tag{6}$$

where $H_*(X, Y)$ is the $\mathbb{Z}$ singular homology of the couple.

We must introduce now the Poincaré polynomial; this algebraic tool allows us to formulate the main theorem of this paragraph.

Definition 8. Let (X, A) be a topological pair. Then the Poincaré polynomial $\mathcal{P}_\lambda(X, A)$ is the formal series in the λ variable with non negative integer coefficients (maybe infinite) defined by

$$\mathcal{P}_\lambda(X, A) = \sum_{q \in \mathbb{N}} \dim H_q(X, A) \lambda^q. \tag{7}$$

Moreover $\mathcal{P}_\lambda(X) := \mathcal{P}_\lambda(X, \emptyset)$.

At last, we can state the so called *Morse relations*, that are useful to estimate the number of critical points of a function.

Theorem 9 (Morse relations)**.** *Let M be a complete C^2 Riemannian manifold, $f \in C^2(M, \mathbb{R})$, and let a, b be two regular values of f. If f_a^b is compact and all the critical points are nondegenerate, then*

$$\sum_{x \, critical \ in \ f_a^b} \lambda^{m(x)} = \mathcal{P}_\lambda(f^a, f^b) + (1 + \lambda)\, \mathcal{Q}_\lambda. \tag{8}$$

If also M is compact, then

$$\sum_{x \, critical} \lambda^{m(x)} = \mathcal{P}_\lambda(M) + (1 + \lambda) \mathcal{Q}_\lambda. \tag{9}$$

where $m(x)$ is the Morse index of f at x, and $\mathcal{Q}_\lambda$ is a formal series with non negative integer coefficients.

If f_a^b is not compact, the above theorem is no longer valid. This assumption can be substituted with the Palais Smale compactness condition, recalled below. This condition permits to extend Morse relations when, as in our case, M is a non compact infinite dimensional manifold.

Definition 10. Let H be an Hilbert space; $f \in C^1(H, \mathbb{R})$ satisfies the $(PS)_c$ condition iff every sequence $\{u_h\}_h \subset H$ s.t.

$$\begin{aligned} &||\nabla f(u_h)|| \to 0; \\ &f(u_h) \to c, \end{aligned}$$

is relatively compact in H.

4 Variational settings

Now let us consider the dynamical system defined by equation (1):

$$\ddot{x} + V'_x(t, x) = 0$$

where $x \in S^1$, $V \in C^2(\mathbb{R} \times S^1)$ and V' denotes the derivative of V with respect to x. We suppose that $V(t, \cdot)$ is T_0-periodic.
We introduce three different spaces:

$$H^1_{k_2T_0} = \{x : x \in H^1_{loc}(\mathbb{R}, S^1),\ x(t + k_2T_0) = x(t)\},$$

the Hilbert space of all the periodic orbits with period k_2T_0, equipped with the following scalar product

$$< u, v >= \frac{1}{k_2T_0} \int_0^{k_2T_0} (\dot{u} \cdot \dot{v} + u \cdot v) dt,$$

with u, $v \in TH^1_{k_2T_0} = H^1_{k_2T_0}$,

$$H^1_{0,k_2T_0} = \{[x] :\ x \in H^1_{loc}(\mathbb{R}, \mathbb{R}),\ x(t + k_2T_0) = x(t)\}$$

the Hilbert space of the periodic orbits with period k_2T_0 in the 0-th connect component, and

$$H^1_{k_1,k_2T_0} = \{[x] :\ x \in H^1_{loc}(\mathbb{R}, \mathbb{R}),\ x(t + k_2T_0) = x(t) + k_1\}$$

the set of k_2T_0 periodic orbits that make k_1 windings in S^1, where $[x] = x$ mod 1. We have that $H^1_{k_1,k_2T_0}$ is an Hilbert affine space. Indeed, given $x(t) \in H^1_{k_1,k_2T_0}$, there exist $y(t) \in H^1_{0,k_2T_0}$ such that

$$x(t) = \frac{k_1}{k_2T_0} t + y(t). \tag{10}$$

We are interested to study the k_2T_0-periodic solution of (1). The equation (1) is the Euler-Lagrange equation corresponding to the functional

$$f(x) = \frac{1}{k_2T_0}\int_0^{k_2T_0} (\frac{1}{2}|\dot{x}|^2 - V(t,x))dt \tag{11}$$

on $H^1_{k_1,k_2T_0}$ and the k_2T_0-periodic solutions of equation (1) are the critical point of the functional (11). It is well known that the functional is C^2 on $H^1_{k_1,k_2T_0}$ and we can apply the Morse theory defining a Morse index for every k_2T_0-periodic solution of (1). If $x(t)$ is a k_2T_0-periodic solutions of equation (1) then

$$f'(x)[y] = \frac{1}{k_2T_0}\int_0^{k_2T_0} (\dot{x}\cdot\dot{y} - V'(t,x)y)dt = 0 \tag{12}$$

for all $y \in H^1_{0,k_2T_0}$ with $y(0) = 0$. The Hessian of the functional f is defined as

$$f''(x)[y][y] = \frac{1}{k_2T_0}\int_0^{k_2T_0} (|\dot{y}|^2 - V''(t,x)y^2)dt \tag{13}$$

and the signature of the Hessian at $x(t)$ is given by the number of negative eigenvalues of (13).

Definition 11. We denote $m(x, k_2T_0)$ the Morse index relative to the k_2T_0 periodic orbit $x(t)$, i.e. the signature of (13).

4.1 Poincaré polynomial of the free loop space

Now we can compute the Poincaré polynomial of the $H^1_{k_1,k_2T_0}$, that is the k_1-th connected component of the path space $H^1_{k_2T_0}$.

We recall some feature of the Poincaré polynomial that we need to prove our result. For all the details and for an exhaustive treatment of the Poincaré polynomial we refer to [3, 4]. Here we recall only the following standard result of algebraic topology.

Remark 12. Let (X, A) and (Y, B) be two pairs of topological spaces. Then

1. if (X, A) and (Y, B) are homotopically equivalent, then $\mathcal{P}_\lambda(X, A) = \mathcal{P}_\lambda(Y, B)$;
2. $\mathcal{P}_\lambda(X \times Y, A \times B) = \mathcal{P}_\lambda(X, A) \cdot \mathcal{P}_\lambda(Y, B)$ (Künnet formula);
3. if x_0 is a single point then $\mathcal{P}_\lambda(\{x_0\}) = 1$; furthermore if X is topologically trivial even $\mathcal{P}_\lambda(X) = 1$;

It is obvious that $H^1_{k_2T_0} \simeq H^1(S^1, S^1)$, and so also $H^1_{k_1,k_2T_0} \simeq H^1_{k_1}(S^1, S^1)$. Furthermore, in order to calculate the Poincaré polynomial of the path space, by the Whitney theorem we know that there is an homotopic equivalence between $H^1(S^1, S^1)$ and $C^0(S^1, S^1)$. This is a standard argument, and can be found, for example, in [8]. Thus, we can consider the k-th connected component of $C^0(S^1, S^1)$ that is the set of continuous maps from S^1 to S^1 with index k (roughly speaking the curves that "turns" k times around S^1). We note this component as $C^0_k(S^1, S^1)$. We want to show the following Lemma

Lemma 13. *For all integer $k_1 \in \mathbb{Z}$, we have that*

$$\mathcal{P}_\lambda(H^1_{k_1,k_2T_0}) = 1 + \lambda$$

Proof. We have just said that $H^1_{k_1,k_2T_0} \simeq H^1_{k_1}(S^1, S^1) \simeq C^0_{k_1}(S^1, S^1)$. Now it's easy to see that

$$\begin{aligned} C^0_{k_1}(S^1, S^1) &\simeq \left\{u : u \in C^0(\mathbb{R}, \mathbb{R}),\ u(t + k_2T_0) = u(t) + k_1\right\} \simeq \\ &\simeq S^1 \times \left\{u : u \in C^0(\mathbb{R}, \mathbb{R}),\ u(t + k_2T_0) = u(t) + k_1,\ u(0) = 0\right\}. \end{aligned}$$

The space $\{u : u \in C^0(\mathbb{R}, \mathbb{R}),\ u(t + k_2T_0) = u(t) + k_1,\ u(0) = 0\}$ is an affine space, so it is contractible and its Poincaré polynomial is equal to 1. Then, by the Künnet formula we obtain

$$\mathcal{P}_\lambda(C^0_{k_1}(S^1, S^1)) = \mathcal{P}_\lambda(S^1) = 1 + \lambda, \tag{14}$$

that concludes the proof. □

4.2 The Palais Smale condition

We show now that the functional

$$f(x) = \frac{1}{T_0} \int \frac{1}{2}|\dot{x}|^2 - V(t, x)\ dt$$

defined at the beginning of this section satisfies the Palais Smale condition. The result is well known because the potential is bounded and we prove it in the standard way.

Proposition 14. *The functional f satisfies the (PS) condition in $H^1_{k_1,k_2T_0}$*

Proof. At first we notice that V is bounded. In fact, V is C^2, periodic in the t variable, and $x(t)$ is periodic. Furthermore, because $x \in H^1_{k_1,k_2T_0}$, is also continuous, so the potential V is bounded.

Suppose that x_n is a Palais Smale sequence, i.e. that

$$f(x_n) = \frac{1}{T_0}\int \frac{1}{2}|\dot{x}_n|^2 - V(t, x_n)\ dt \to c \in \mathbb{R}; \tag{15}$$

$$f'(x_n)[v] = \frac{1}{T_0}\int \dot{x}_n\dot{v} - V'(t, x_n)v\ dt \to 0\ \forall v \in H^1_{0,k_2T_0}. \tag{16}$$

By (15), we know that $f(x_n)$ is bounded. Because $V(t, x_n)$ is bounded, we have that also $||x_n||_{H_1}$ is bounded. Thus, up to subsequence, $x_n \rightharpoonup x$ weakly in H^1, furthermore, for the Sobolev immersion theorem, we have that $x_n \to x$ in L^2 and uniformly. By (16) we have that

$$f'(x_n)[x_n - x] = \frac{1}{T_0}\int < \dot{x}_n, \dot{x}_n - \dot{x} > -V'(t, x_n)(x_n - x)\ dx \to 0. \tag{17}$$

We know that $V'(t, x_n) \to V'(t, x)$ uniformly (and thus L^2). Then

$$\int V'(t, x_n)(x_n - x) \to 0. \tag{18}$$

So we obtain that

$$\frac{1}{T_0}\int < \dot{x}_n, \dot{x}_n - \dot{x} >= \frac{1}{T_0}\int |\dot{x}_n|^2 - \frac{1}{T_0}\int \dot{x}_n\dot{x} \to 0. \tag{19}$$

But, because $x_n \to x$ weakly in H^1 we have that

$$\int \dot{x}_n\dot{x} \to \int |\dot{x}|^2, \tag{20}$$

so we have that

$$||x_n||_{H^1} \to ||x||_{H^1}, \tag{21}$$

that concludes the proof. $\square$

5 The Bott and Maslov indexes

The Morse index of a periodic orbit is strongly related to two others indexes. One is the Maslov index and the other is an index that we have called Bott index since it has been introduced in the study of geodesics by Bott.

These indexes turn out to have the same numerical value but they refer to different mathematical objects. Indeed, the Morse index of a periodic orbit $x(t)$ measures the signature of the Hessian of f at $x(t)$, the Maslov index the half windings in the symplectic group $Sp(2)$ of the matrix of the

fundamental solutions of the linearized equation around $x(t)$ and the Bott index the negative eigenvalues of the operator $-\ddot{y} - V''(x(t),t)y$ associated to the linearized equation.

We need to introduce the Bott index to easily compute the twisting frequency of a periodic orbit while the Maslov index to characterize the periodic orbits of type α and type β.

5.1 The Bott index and the twisting frequency

Let us consider, for $\sigma \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$,

$$L^2_{\sigma,T} = \{x \in L^2_{loc}(\mathbb{R}, \mathbb{C}^{\mathbb{N}}) \ : \ x(t+T) = \sigma \cdot x(t) \text{ for a.a. } t \in \mathbb{R}\}$$

where L^2_{loc} is the set of function $x : \mathbb{R} \to \mathbb{C}^{\mathbb{N}}$ which are measurable and whose square is locally integrable. $L^2_{\sigma,T}$ is an Hilbert space with the following scalar product

$$(x,y) = \frac{1}{T}\int_0^T (x(t), y(t))_{\mathbb{C}^{\mathbb{N}}} dt.$$

Now, we consider the following differential equation

$$\ddot{y} + A(t)y = -\lambda y, \tag{22}$$

with $y \in \mathbb{C}^{\mathbb{N}}, \lambda \in \mathbb{R}$ and $A(t)$ a family of real symmetric $N \times N$ matrices T_0-periodic, defined on L^2_{σ,T_0}.
Let $W^2_{loc}(\mathbb{R}, \mathbb{C}^{\mathbb{N}})$ be the space of functions having two square locally integrable derivative and $\mathcal{L}_{\sigma,T_0}$ be the extension to $W^2_{loc}(\mathbb{R}, \mathbb{C}^{\mathbb{N}}) \cap L^2_{\sigma,T_0}$ of the operator

$$-\ddot{y} - A(t)y.$$

The eigenvalue problem (22) becomes

$$\mathcal{L}_{\sigma,T_0} y = \lambda y, \tag{23}$$

with $y \in W^2_{loc}(\mathbb{R}, \mathbb{C}^{\mathbb{N}}) \cap L^2_{\sigma,T_0}$. The spectrum of this selfadjoint unbounded operator is discrete with a finite number of negative eigenvalue.
This fact allows us to define a function

$$j(T_0, \cdot) : S^1 \to \mathbb{N}$$

as follows:

$$j(T_0,\sigma) = \left\{ \begin{array}{c} \text{number of negative eigenvalues of } \mathcal{L}_{\sigma,T_0} \\ \text{counted with their multiplicity.} \end{array} \right\} \tag{24}$$

In order to define the Bott index we need that the operator $\mathcal{L}_{1,T_0}$ is nondegenerate, i.e that 0 is not an eigenvalue of $\mathcal{L}_{1,T_0}$.

In this case we can define the Bott index in the following way:

Definition 15. We denote the function $j(T_0, 1)$ the Bott index relative to the equation $\ddot{y} + A(t)y = 0$ in the interval $[0, T_0]$.

Now let $W(t) : \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ the matrix of the fundamental solutions relative to the equation $\ddot{y} + A(t)y = 0$, namely the solution of the following Cauchy problem

$$\begin{cases} \dot{W}(t) + \mathcal{A}(t)W(t) = 0 \\ W(0) = I. \end{cases}$$

where

$$\mathcal{A}(t) = \begin{pmatrix} 0 & I \\ -A(t) & 0 \end{pmatrix}.$$

The eigenvalues of $W(T_0)$ are called Floquet multipliers. The nondegenerate condition means that the linear system $\ddot{y}+A(t)y = 0$ does not have any non-trivial T_0-periodic solutions, i.e that 1 is not a Floquet multiplier of $W(T_0)$. The Bott index fulfills the following properties

Proposition 16. *The function $j(T_0, \sigma)$ satisfies the following properties.*
(i) $j(T_0, \sigma) = j(T_0, \bar{\sigma})$

(ii) if $j(T_0, \sigma)$ is discontinuous at the point σ^* then σ^* is a Floquet multiplier

(iii) $|j(T_0, \sigma_2) - j(T_0, \sigma_1)| \leq l \quad \forall \sigma_2, \sigma_1 \in S^1 - \{+1, -1\}$ *where $2l$ is the number of non-real Floquet multipliers on S^1 counted with their molteplicity*

(iv)

$$j(kT_0, \theta) = \sum_{j=0}^{k-1} j(T_0, \sigma_j)$$

where $\sigma_0, \sigma_1, ..., \sigma_{k-1}$ are the k values of $\sqrt[k]{\theta}$.

The proof of (i), (ii), (iii), (iv) is contained in [3].

The Bott index allows to define the twisting frequency as follows:

$$\tau = \frac{1}{2\pi T_0} \int_0^{2\pi} j(T_0, exp[i\omega])d\omega. \tag{25}$$

Proposition 17. *The twisting frequency satisfies the following properties:*
(i) $\tau = \lim_{T\to\infty} \frac{1}{T} j(T, 1) \quad T = kT_0$

(ii) $\tau = \frac{1}{2\pi T} \int_{S^1} j(T, \sigma)d\sigma \quad T = kT_0$

(iii) $|T\tau - j(T,\sigma)| \leq l \quad \forall\sigma \in S^1 - \{+1,-1\}$ *where* $2l$ *is the number of non-real Floquet multipliers on* S^1 *counted with their multiplicity and* $T = kT_0$

(iv) $\forall\sigma \in S^1$ *we have* $\tau = \lim_{T\to\infty} \frac{1}{T} j(T,\sigma) \quad T = kT_0$

The proof of (i), (ii), (iii), (iv) is contained in [3].

5.2 The Maslov index and the geometrical representation of $Sp(2)$

In this section we give some properties of the Morse index by means of Maslov index in the two dimensional case.

Let us consider the linear equation

$$\ddot{y} + A(t)y = 0 \tag{26}$$

where $A(t)$ is T_0-periodic. Let $W(t)$ be the matrix of the fundamental solutions of the linear equation (26) at time t, with $t \in [0,T]$. The matrix $W(t)$ is unimodular, i.e it is symplectic and we can associate to the linear equation (26) a path γ in the symplectic group. The Maslov index is an integer associated to the path of $W(t)$ in the symplectic group. The Maslov index theory for any non degenerate path in $Sp(2)$ was established first in [7] and [11]; we avoid rigorous definitions for the sake of brevity and we refer to the book of Abbondandolo [1].

Loosely speaking, the Maslov index is the number of half windings made by the path in $Sp(2)$. However, in order to give a geometrical meaning of the Maslov index we need to describe some properties of the symplectic group of the plane.

The symplectic group of the plane $Sp(2)$ consists of the real matrices two by two A such that $A^T JA = J$, where A^T is the transpose of A and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues λ_1 and λ_2 of $A \in Sp(2)$ are of the following form:

- $\lambda_1 = \lambda_2 = 1$
- $\lambda_1 = \lambda_2 = -1$
- $\lambda_1 = \bar{\lambda_2} \quad \lambda_1, \lambda_2 \in S^1 - \{+1,-1\}$

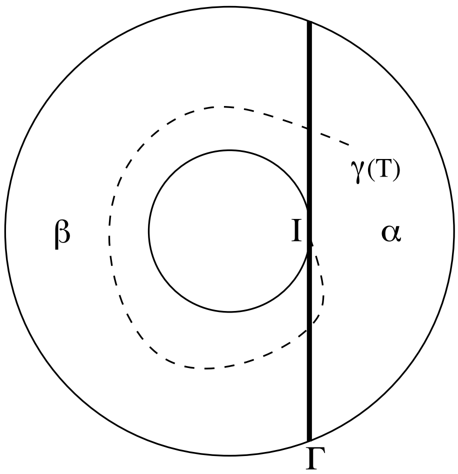

Figure 1: The sets α, β and the set Γ of degenerate matrices. The set Γ is represented by the vertical line.

- $\lambda_1 = \frac{1}{\lambda_2} \quad \lambda_1, \lambda_2 \in \mathbb{R} - \{+1, -1\}$

A parametrization of $Sp(2)$ due to Gel'fand and Lidskiĭ allows to visualize the simplectic group as $S^1 \times D$ where D is the unitary disk.

The set of matrices in $Sp(2)$ that correspond to the degenerate condition, i.e those such that 1 is a Floquet multiplier, disconnect the simplectic group into two regions α and β. The set α is that of the matrices with distinct and real positive Floquet multipliers and β is that of matrices with complex or real negative Floquet multipliers. Figure 1 gives a rough idea of sets α and β in the symplectic group; we refer to [1] for a rigorous and pretty picture.

Now, we can state the proposition that relates the Bott index with the Maslov index and that characterize the periodic orbit depending on the parity of the Maslov index.

Proposition 18. *The Maslov index $\mu_\gamma(T)$ of the path $\gamma : [0, T] \to Sp(2)$ fulfills the following properies*

(i) $\mu_\gamma(T) = j(T, 1)$

(ii) $\mu_\gamma(T)$ *is even if and only if the Floquet multipliers of* $\gamma(T)$ *are distinct and real positive*

(iii) $\mu_\gamma(T)$ *is odd if and only if the Floquet multipliers of* $\gamma(T)$ *are complex or real negative*

Proof. The proof of (i), (ii), (iii) can be found in [2], [10] and [1] $\square$

6 Main Results on periodic orbits

We want to introduce the rotation frequency of a curve as follows.

Definition 19. If $x(t) \in H^1_{k_1,k_2T_0}$, then its rotation frequency is

$$\rho(x) = \frac{k_1}{k_2 T_0} \tag{27}$$

We recall that the Morse index allows to separate the periodic orbits in two distinct classes as described in the previous section.

Definition 20. Let $x(t)$ be a periodic solution of (1) in $H^1_{k_1,k_2T_0}$; $x(t)$ is periodic of type α (positive distinct Floquet multipliers) if $m(x, k_2T_0)$ is even, and periodic of type β if $m(x, k_2T_0)$ is odd (complex or negative Floquet multipliers).

Proposition 21. *For any* x *periodic solution in* $H^1_{k_1,k_2T_0}$*, we have that* $f(x)$ *is bounded by a constant which depends only on* k_1 *and* k_2T_0*.*

Proof. The value of $|V'(t, x(t))|$ is bounded because $V'(t, x)$ is a C^1 function on the compact set $S^1 \times S^1$. Thus, it is sufficient to prove that $|\dot{x}(t)|$ is bounded for all x periodic solutions in $H^1_{k_1,k_2T_0}$. By eq. (10), we know that $x(t) = \rho t + y(t)$ where $y \in H^1_{0,k_2T_0}$.

Moreover, we have that

$$\int_0^{k_2T_0} \dot{y}(t)dt = y(k_2T_0) - y(0) = 0,$$

therefore, for any $y \in H^1_{0,k_2T_0}$, there exist $\bar{t}$ such that $\dot{y}(\bar{t}) = 0$. For the Lagrange theorem there exist $\xi \in [0, k_2T_0]$ such that

$$\frac{|\dot{y}(t) - \dot{y}(\bar{t})|}{|t - \bar{t}|} = |\ddot{y}(\xi)| = |\ddot{x}(\xi)| = |V'_x(\xi, x(\xi))| \leq C. \tag{28}$$

So, $|\dot{y}(t)| \leq C|t-\bar{t}| \leq Ck_2T_0$. Finally, for any periodic solution $x(t) \in H^1_{k1,k_2T_0}$

$$|\dot{x}(t)| = |\rho + \dot{y}(t)| \leq \rho + Ck_2T_0.$$

$\square$

Proposition 22. *The number of critical point of f is even in $H^1_{k_1,k_2T_0}$.*

Proof. The functional f is bounded on the periodic solutions in $H^1_{k_1,k_2T_0}$ by the above proposition. The Palais-Smale condition and the assumption that eq. (1) is non-resonant assures that the critical point are in a finite number.

We apply the Morse relations given by (9)

$$\sum_{x\text{critical}} \lambda^{m(x)} = \mathcal{P}_\lambda(M) + (1+\lambda)\mathcal{Q}_\lambda.$$

where M is $H^1_{k_1,k_2T_0}$. The lemma (13) shows that $\mathcal{P}_\lambda(H^1_{k_1,k_2T_0}) = 1+\lambda$ and, therefore, the Morse relation becomes

$$\sum_{x\text{critical}} \lambda^{m(x)} = 1+\lambda+(1+\lambda)\mathcal{Q}_\lambda = (1+\lambda)\tilde{\mathcal{Q}}_\lambda. \tag{29}$$

$\lambda = 1$ shows that the number of periodic solutions are $2\tilde{\mathcal{Q}}_\lambda$. □

Definition 23. Let $\rho = \frac{k_1}{k_2T_0}$. We set

$$\begin{aligned} n_\alpha(\tau,\rho) &= \left\{ \begin{array}{c} \text{number of fundamental solutions } x \text{ of type } \alpha \text{ with} \\ \tau(x)=\tau;\ \rho(x)=\rho \end{array} \right\} \\ n_\beta(\tau,\rho) &= \left\{ \begin{array}{c} \text{number of fundamental solutions } x \text{ of type } \beta \text{ with} \\ \tau(x)=\tau;\ \rho(x)=\rho \end{array} \right\} \end{aligned}$$

and the function

$$\chi(\tau,\rho) \ := \ n_\alpha(\tau,\rho) - n_\beta(\tau,\rho) \tag{30}$$

Remark 24. For the periodic solutions with the Morse index equal to an even number $2m$ we have

$$\tau = \frac{2m}{k_2T_0}$$

while for the solutions with Morse index equal to $2m+1$ we have

$$\tau \in \left(\frac{2m}{k_2T_0}, \frac{2m+2}{k_2T_0}\right).$$

The Proposition 18 implies that a periodic solution $x(t)$ is periodic of type α iff the Floquet exponent are distinct and real positive, i.e iff the symplectic matrix $\gamma(T)$ is in the α component of $Sp(2)$. On the contrary, $x(t)$ is periodic of type β iff the eigenvalues are complex or real negative, i.e. if $\gamma(T)$ is in the β component of $Sp(2)$. The periodic orbits with an even Morse index are of kind α, i.e have positive distinct Floquet multipliers. By Proposition 17

we have that the twisting frequency is the mean Morse index, so if $x(t)$ is of kind α we have

$$\tau = \frac{1}{2\pi T}\int_{S^1} j(T,\sigma)d\sigma = \frac{1}{2\pi k_2 T_0}\int_{S^1} 2m\ d\sigma = \frac{2m}{k_2 T_0}.$$

When the Morse index is an odd number we have that $j(T,\sigma)$ is not constant but it assumes only the values $2m+1$ and $(2m+1)\pm 1$, so we obtain the other estimate.

Proposition 25. *Let $\rho = \frac{k_1}{k_2 T_0}$. We have $\chi(0,\rho) > 0$.*

Proof. For all $\rho = \frac{k_1}{k_2 T_0}$, by the previous remark we have immediately that $\chi(\tau,\rho) \geq 0$ for $\tau = \frac{2m}{k_2 T_0}$ and $\chi(\tau,\rho) \leq 0$ for $\tau \in \left(\frac{2m}{k_2 T_0}, \frac{2m+2}{k_2 T_0}\right)$. Furthermore, the Morse relations given by (9) and Lemma 13 show that there exist periodic orbits with Morse index 0. This concludes the proof. □

Definition 26. For all $\rho = \frac{k_1}{k_2 T_0}$ we set the multiplicity functions

$$\nu(\tau,\rho) := \sum_{\zeta \leq \tau} \chi(\zeta,\rho)$$

$$\eta(\tau,\rho) := \sum_{\zeta < \tau} \chi(\zeta,\rho)$$

Clearly $\nu(\tau,\rho) = \eta(\tau,\rho) + \chi(\tau,\rho)$.
These functions are well defined because $\chi(\tau,\rho) \neq 0$ only for a finite number of τ. The total number of α and β periodic solutions are given by the Morse relations given by (9). The topology of $H^1_{k_1,k_2 T_0}$ given by Lemma 13 and the Morse relations imply that the total number of solutions with even Morse index are equal to the number of solutions with odd Morse index. If we call τ_{max} the maximum value of τ among the fundamental periodic solutions, we have $\nu(\tau,\rho) = 0$ if $\tau \geq \tau_{max}$.

Proposition 27. *Let $\rho = \frac{k_1}{k_2 T_0}$, there exists $\epsilon_0 > 0$ such that*

$$\begin{aligned} \eta(\tau+\epsilon,\rho) &= \nu(\tau,\rho) \qquad \forall\ 0 < \epsilon < \epsilon_0; \\ \nu(\tau+\epsilon,\rho) &= \nu(\tau,\rho) \qquad \forall\ 0 < \epsilon < \epsilon_0. \end{aligned}$$

Proof. Given $\rho = \frac{k_1}{k_2 T_0}$, we know that there exist a finite number of fundamental solutions. Therefore, there exists $\epsilon_0 > 0$ such that $\chi(\xi,\rho) = 0$ if $\xi \in (\tau, \tau+\epsilon_0)$. The proof follows straightforward. □

Proposition 28. *Let y be a non-fundamental periodic solution of $\ddot{x}+V'(t,x)=0$ in $C^2_{pk_1,pk_2T_0}$ with p prime. Then $y(t)$, $y(t+k_2T_0)$, $y(t+2k_2T_0)$,..., $y(t+(p-1)k_2T_0)$ are p distinct non-fundamental periodic solutions.*

Proof. y is a periodic solution that makes pk_1 windings in pk_2T_0 time; y is non-fundamental, thus, it is nonperiodic of period k_2T_0.

At first we show that $y(t+lk_2T_0)$ is a solution. We have that

$$\begin{aligned} \ddot{y}(t+lk_2T_0)+V'(t,y(t+lk_2T_0)) &= \\ =\ddot{y}(t+lk_2T_0)+V'(t+lk_2T_0,y(t+lk_2T_0)) &= 0. \end{aligned}$$

Furthermore, suppose that there exists $l\neq m$ with $l,m<p$ such that

$$y(t+lk_2T_0)=y(t+mk_2T_0).$$

After a change of variables we have that

$$y(t+(l-m)k_2T_0)=y(t)\ \forall t.$$

but p is prime and that contradicts our hypothesis □

Now, we can prove Theorem 3 and Theorem 4.

Proof of Theorem 3. Given any rotation frequency $\rho\in\frac{1}{T_0}\mathbb{Q}$, we take k_1 and k_2 coprime such that $\rho=\frac{k_1}{k_2T_0}$. If eq. (1) is non-resonant we have, by Proposition 22, an even number of periodic solutions in $H^1_{pk_1,pk_2T_0}$ for any $p\in\mathbb{N}^+$.

These periodic solutions are fundamental solutions if we take $p=1$.

Clearly, if $x\in H^1_{k_1,k_2T_0}$ then $x\in H^1_{pk_1,pk_2T_0}$ and the Morse index $m(x,pk_2T_0)$ fulfills the property (iii) of Proposition 17

$$\tau(x)\,pk_2T_0-1\leq m(x,pk_2T_0)\leq\tau(x)\,pk_2T_0+1.$$

The Morse relations assures that there exist $y\in H^1_{pk_1,pk_2T_0}$ with $m(y,pk_2T_0)=1$. This orbit fulfills $\rho(y)=\frac{k_1}{k_2T_0}$ and it is non-fundamental when p is sufficently large because it cannot be k_2T_0 periodic. Indeed, the property (iii) of Proposition 17 assures that all the periodic orbits $x(t)$ in $H^1_{k_1,k_2T_0}$ with Morse index 1 have a Morse index $m(x,pk_2T_0)>1$ when p is sufficently large.

Moreover, the periodic orbits $x(t)$ in $H^1_{k_1,k_2T_0}$ with Morse index 0 have a Morse index $m(x,pk_2T_0)=0$ for the same reason.

This proves that, taken p sufficently large, the periodic orbit $y\in H^1_{pk_1,pk_2T_0}$ with $m(y,pk_2T_0)=1$ cannot be k_2T_0 periodic and therefore it is non-fundamental.

Hence, there exist infinitely many non-fundamental orbits with $\rho=\frac{k_1}{k_2T_0}$. □

Proof of Theorem 4. Without any lack of generality we demonstrate the theorem for $k_1 = k$ and $k_2 = 1$. The generalization to $\rho = \frac{k_1}{k_2 T_0}$ is straightforward. We consider, therefore, the case $\rho = \frac{k}{T_0}$. Moreover, in order to avoid a too heavy notation we will use $\nu(\tau)$, $\chi(\tau)$ and $\eta(\tau)$ instead of $\nu(\tau, \rho)$, $\chi(\tau, \rho)$ and $\eta(\tau, \rho)$. All this functions have to be considered, however, depending on ρ. The leading idea for these results is that a T_0 periodic solution $x(t) \in H^1_{k,T_0}$ is also a pT_0 periodic solution. In this case we can consider $x \in H^1_{pk,pT_0}$.

The Morse relations (9) for the pT_0-periodic solutions may be written in the following way

$$\sum_j a_j \lambda^j = 1 + \lambda + (1 + \lambda)\mathcal{Q}_\lambda = (1 + \lambda) \sum_j q_j \lambda^j.$$

with a compact notation

$$\begin{aligned} a_0 &= q_0 \\ a_j &= q_j + q_{j-1} \end{aligned} \tag{31}$$

or in a non compact form

$$\begin{aligned} q_0 &= a_0 \\ q_1 &= a_1 - a_0 \\ q_2 &= a_2 - a_1 + a_0 \\ \ldots &= \ldots\ldots\ldots \\ q_{2n} &= a_{2n} - \ldots\ldots\ldots - a_1 + a_0 \\ q_{2n+1} &= a_{2n+1} - \ldots\ldots\ldots + a_1 - a_0 \end{aligned} \tag{32}$$

Let us consider the Modular arithmetic given by the function $[\cdot] : \mathbb{Z} \to \mathbb{Z}_p$,. For any a_j, Proposition 28 implies that

$$[a_j] = [\alpha_j]$$

where α_j is the number of the pT_0-periodic solutions with Morse index j that are fundamental solutions.
If j is even, the α_j fundamental solutions $x(t)$ are of kind α and we have

$$j = pm(x, T_0) = p\tau(x)\, T_0.$$

We have

$$\tau(x) = \frac{j}{pT_0}$$

and

$$\alpha_j = \chi(\tau(x)) = \chi\left(\frac{j}{pT_0}\right).$$

If j is odd, the α_j fundamental periodic solutions $x(t)$ are of kind β and we have

$$\tau(x)\,pT_0 - 1 < j < \tau(x)\,pT_0 + 1$$

and, hence,

$$\frac{j-1}{pT_0} < \tau(x) < \frac{j+1}{pT_0}$$

$$\tau(x) \in \left(\frac{j-1}{pT_0}, \frac{j+1}{pT_0}\right) = \frac{1}{pT_0}\,(j-1, j+1)\,.$$

We obtain

$$\alpha_j = -\sum_{\tau\in\left(\frac{j-1}{pT_0},\frac{j+1}{pT_0}\right)} \chi(\tau)$$

If we take p prime and we use the Modular arithmetics, the Morse relations (32) becomes

$$\begin{aligned}
[q_0] &= [\alpha_0] = \chi(0) = \nu(0)\\
[q_1] &= [\alpha_1] - [\alpha_0] = \left[-\sum_{\tau\in\left(\frac{0}{pT_0},\frac{2}{pT_0}\right)} \chi(\tau)\right] - [\chi(0)] = \left[\chi\left(\frac{2}{pT_0}\right) - \nu\left(\frac{2}{pT_0}\right)\right]\\
[q_2] &= [\alpha_2] - [\alpha_1] + [\alpha_0] = \left[\chi\left(\frac{2}{pT_0}\right)\right] - \left[-\sum_{\tau\in\left(\frac{0}{pT_0},\frac{2}{pT_0}\right)} \chi(\tau)\right] + [\chi(0)]\\
&= \left[\nu\left(\frac{2}{pT_0}\right)\right]\\
\ldots &= \ldots\ldots\ldots\\
[q_{2n}] &= [\alpha_{2n}] - \ldots\ldots\ldots - [\alpha_1] + [\alpha_0] = \left[\nu\left(\frac{2n}{pT_0}\right)\right]\\
[q_{2n+1}] &= [\alpha_{2n+1}] - \ldots\ldots\ldots + [\alpha_1] - [\alpha_0] = \left[\chi\left(\frac{2n+2}{pT_0}\right) - \nu\left(\frac{2n+2}{pT_0}\right)\right].
\end{aligned}$$

The periodic solutions with Morse index $2n$ are of type α and with twisting frequency $\tau = \frac{2n}{pT_0}$.

We have

$$[q_{2n}] = \left[\nu\left(\frac{2n}{pT_0}\right)\right].$$

If $[\nu\left(\frac{2n}{pT_0}\right)] \neq 0$, we have $[q_{2n}] \neq 0$ and, therefore, $q_{2n} \neq 0$ and $a_{2n} \geq [\nu\left(\frac{2n}{pT_0}\right)]$.

On the other hand, the periodic solutions with twisting frequency τ such that $|\tau - \frac{2n+1}{pT_0}| < \frac{1}{pT_0}$ are of type β with Morse index $2n+1$.

We have

$$[q_{2n+1}] = \left[\chi\left(\frac{2n+2}{pT_0}\right) - \nu\left(\frac{2n+2}{pT_0}\right)\right] = [-\eta(\frac{2n+2}{pT_0})].$$

If $[-\eta(\frac{2n+2}{pT_0})] \neq 0$, we have $[q_{2n+1}] \neq 0$ and, therefore, $q_{2n+1} \neq 0$ and $a_{2n+1} \geq [-\eta(\frac{2n+2}{pT_0})]$.

We have demonstrated that for all $\tau \in \mathbb{N}/(pT_0)$, p prime, there exist orbits with twisting frequency arbitrary close to τ.

In particular, if $\tau = \frac{2n}{pT_0}$, there exist at least $[\nu(\tau)]$ solutions $x(t)$ of type α and pT_0-periodic such that

$$\tau(x) = \tau$$

On the other hand, if $\tau = \frac{2n+1}{pT_0}$, there exist at least $[-\eta(\frac{2n+2}{pT_0})]$ solutions $y(t)$ of type β and pT_0-periodic such that

$$|\tau(y) - \tau| < \frac{1}{pT_0}.$$

□

Remark 29. Theorem 4 gives a lower bound on the number of solutions in C_{pk_1,pk_2T_0}. Notice that the periodic solutions are non-fundamental any time we choose τ sufficently far from any twisting number of the fundamental solutions. In this case, the non-fundamental periodic solutions are at least p by Proposition (28).

As a consequence of Theorem 4 we can prove the following corollary.

Corollary 30. *Let $\rho = \frac{k_1}{k_2T_0}$ and τ such that $\nu(\tau,\rho) \neq 0$.*
Then, there exist a sequence of non-fundamental orbits x_n of type α and a sequence of non-fundamental orbits y_n of type β such that

$$\tau(x_n) \to \tau$$

$$\tau(y_n) \to \tau.$$

Proof. Given any τ, we can choose two approximations of τ of the following form: $\tau_\alpha = \frac{2n}{pk_2T_0} > \tau$ and $\tau_\beta = \frac{2n+1}{pk_2T_0} > \tau$.

We know, by Proposition 27, that $\nu(\tau_\alpha,\rho) = \nu(\tau,\rho) \neq 0$ and $\eta(\tau_\beta,\rho) = \eta(\tau,\rho) \neq 0$ when τ_α and τ_β are sufficiently close to τ, i.e definitely for p

large. Therefore, we can choose p such that $\nu(\tau_\alpha, \rho) \neq 0 \pmod p$ and $-\eta\left(\frac{2n+2}{pk_2T_0}, \rho\right) \neq 0 \pmod p$. By Theorem 4, we have at least $\nu(\tau_\alpha, \rho) \pmod p$ orbits with twisting frequency τ_α and $-\eta\left(\frac{2n+2}{pk_2T_0}, \rho\right) \pmod p$ orbits with twisting frequency close to τ_β.

Thus, if we take a sequence of τ_α and τ_β which converges to τ we find a sequence of orbits with twisting frequency that converges to τ. We can choose these orbits to be non-fundamental because the fundamental orbits are in a finite number. $\square$

By this corollary we can prove the last result claimed in the introduction.

Proof of Corollary 5. For any $(\tau, \rho) \in \Sigma$, we can choose a sequence $\rho_k \to \rho$ and a sequence $\tau_k \to \tau$ such that, for all k, $\nu(\tau_k, \rho_k) \neq 0$. So, by the previous corollary, we can find two sequence of non-fundamental orbits x_n^k and y_n^k such that

$$\begin{aligned} \tau(x_n^k) &\to \tau_k \qquad \rho(x_n^k) \to \rho_k; \\ \tau(y_n^k) &\to \tau_k \qquad \rho(y_n^k) \to \rho_k. \end{aligned}$$

A diagonal argument proofs the corollary. $\square$

Proposition 31. *For all $\rho = \frac{1}{T_0}\mathbb{Q}$, let x_n be a sequence of non-fundamental orbits such that $\tau(x_n) \to \tau$, then $x_n \to x$ in C^1_{loc}.*

Proof. Let $\rho = \frac{k_1}{k_2T_0}$ and x_n be the non-fundamental orbits with $\rho(x_n) = \rho$ and $\tau(x_n) \to \tau$. The orbit x_n makes $k_{1,n}$ windings of S^1 in $k_{2,n}$ time, with $\frac{k_{1,n}}{k_{2,n}T_0} = \rho$.

The orbits x_n are solution of eq.(1) and $\ddot{x}_n$ is bounded by the maximum value of $|V'(t, x_n(t))|$ which is a C^1 function on the compact set $S^1 \times S^1$. In order to prove that $x_n \to x$ in C^1_{loc} we want to show that, fixed a finite interval of time $I = [0, D]$, $x_n \in W^{2,\infty}(I)$. It is sufficient to prove that $\dot{x}_n(0)$ is bounded. Indeed,

$$\dot{x}_n(t) = \dot{x}_n(0) + \int_0^t -V'(x(s), s)ds.$$

The right-hand side is bounded in I iff $\dot{x}_n(0)$ is bounded.
By eq. (10), $x_n(t) = \rho t + y_n(t)$ with $y_n \in H^1_{0,k_{2,n}T_0}$. We have that $\dot{x}_n(t) = \rho + \dot{y}_n(t)$. The function $y_n(t)$ is periodic, therefore $\dot{y}_n(\xi_n) = 0$ for $\xi_n \in [0, k_{2,n}T_0]$. Proposition 28 shows that $x_n(t)$, $x_n(t+k_2T_0)$,..., are distinct non-fundamental periodic orbits with the same τ. We can shift the orbits such a way that $\xi_n \in [0, k_2T_0]$.

All the orbits x_n have a point $\xi_n \in [0, k_2 T_0]$ where the derivative is zero, therefore they should have bounded initial velocity by means of the Lagrange theorem. Indeed

$$\left| \frac{\dot{y}_n(\xi_n) - \dot{y}_n(0)}{k_2 T_0} \right| \leq \left| \frac{\dot{y}_n(\xi_n) - \dot{y}_n(0)}{\xi_n} \right| \leq const.$$

Thus, $x_n \in W^{2,\infty}(I)$ which is embedded with a compact embedding in $C^1(I)$. $\square$

The authors would like to express thanks to Alberto Abbondandolo for fruitful discussions in the preparation of the paper.

References

[1] Alberto Abbondandolo, *Morse theory for Hamiltonian systems*, Pitman Reaserch Notes in Mathematics, Chapman & Hall, London , 2001

[2] Tianquing An, Yiming Long, *Index theories of second order Hamiltonian systems* Nonlinear Anal. T.M.A **34** (1998), 585-592

[3] Vieri Benci, *A new approach to the Morse-Conley theory and some applications*, Annali di Matematica Pura ed Applicata (IV) **158** (1991), 231–305.

[4] Vieri Benci, *Morse theory for C^1-functionals and Conley blocks*, Topological Methods in Nonlinear Analysis **4** (1994), 365–398.

[5] Vieri Benci, *Introduction to Morse theory. A new approach*, Topological Nonlinear Analysis: Degree, Singularity, and Variations (Michele Matzeu and Alfonso Vignoli, eds.), Progress in Nonlinear Differential Equations and their Applications, no. 15, Birkhäuser, Boston, 1995, pp. 37–177.

[6] Raoul Bott, *Lectures on Morse theory, old and new*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 2, 331–358.

[7] Charles Conley and Eduard Zehnder, *Morse-type Index Theory for Flows and Periodic Solutions for Hamiltonian Equations*, Communications on Pure and Applied Mathematics, Vol. XXXVII, (1984), 207-253

[8] Wilhelm Klingenberg, *Lectures on closed geodesics*, Springer-Verlag, Berlin, 1978, Grundlehren der Mathematischen Wissenschaften, Vol. 230.

[9] Yiming Long, *Multiple periodic points of the Poincare map of Lagrangian systems on tori*, Math. Z. **233** (2000), no. 3, 443–470

[10] Yiming Long, Tianquing An, *Indexing the domains of instability for Hamiltonian systems* NoDEA **5** (1998) 461–478

[11] Yiming Long, Eduard Zehnder, *Morse theory for forced oscillations of asymptotically linear Hamiltonian systems*, Stochastic Process, Physics and Geometry, S. Albeverio et al. Ed., World Scientific, Singapore, 1990, 528–563

[12] Richard S. Palais, *Morse theory on Hilbert manifolds*, Topology **2** (1963), 299–340.

[13] Paul H. Rabinowitz, *Variational methods for Hamiltonian systems.* Handbook of dynamical systems, Vol. 1A, 1091–1127, North-Holland, Amsterdam, 2002.

[14] Dietmar Salamon, Eduard Zehnder, *Morse theory for periodic solutions of Hamiltonian systems and the Maslov index*, Comm. Pure and Appl. Math. **45** (1992), 1303–1360

[15] Enrico Serra, Massimo Tarallo, Susanna Terracini, *Subharmonic solutions to second-order differential equations with periodic nonlinearities*, Nonlinear Anal. Ser. A: Theory Methods **45** (2000), 649–667